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Current Distributions in a Two-Dimensional Random-Resistor Network

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The current and logarithm-of-the-current distributions $\hat{n}(|i|)$ and $n(|\ln |i||)$ on bond diluted two-dimensional random-resistor networks at the percolation threshold are studied by a modified transfer matrix method. The kth moment $(-9 \leqslant k \leqslant 8)$ of $n(|\ln |i||)$ i.e., $\langle |\ln |i||^k \rangle$, is found to scale with the linear size L as $(\ln L)^{\beta(k)}$. The exponents $\beta(k)$ are not inconsistent with the recent theoretical prediction $\beta(k) = k$, with deviations which may be attributed to severe finite-size effects. For small currents, $\ln n(y) \approx -\gamma y$, yielding information on the threshold below which the multifractality of $\hat{n}(|i|)$ breaks down. Our numerical results for the moments of the currents are consistent with other available results.

KEY WORDS: Percolation; multifractals; transport processes; distribution.

1. INTRODUCTION

Multifractal distributions have been the center of much recent research. A specific example on which many of the general questions have been studied in detail concerns the distribution of currents on percolating resistor networks.[1–7] Consider a two-dimensional square lattice network of size $L \times L$, where each bond has an Ohmic conductance $\sigma = 1$ with probability p, or $\sigma = 0$ with probability $(1-p)$. In what follows we consider only the percolation threshold $p_c = 1/2$ when the percolating spanning cluster which

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connects two opposite edges of the network is a fractal. We next pass a unit current between two parallel busbars on these opposite edges. Denoting the absolute value of the current in the bond b of the network by i_b, one considers the distribution $\hat{n}(i_b)$ of these currents and its unnormalized moments

$$\hat{M}_q = \sum_b |i_b|^{2q} = \int di_b \hat{n}(i_b)\,|i_b|^{2q} \tag{1.1}$$

For $q < 0$, the sum contains only terms with $i_b \neq 0$.

Multifractal behavior implies that asymptotically, for large L, one has the power law behavior

$$\hat{M}_q \approx A_q L^{\tilde{\psi}(q)} \tag{1.2}$$

[$\tilde{\psi}(q)$ is sometimes[1] denoted by $-x_q$].

If (instead of a unit current) one applies a unit voltage between the busbars, then each current i_b is replaced by $i'_b = i_b/R$, where $R = \hat{M}_1$ is the net resistance between the busbars. As a result, the unnormalized moments of the currents behave asymptotically as

$$\hat{M}^v_q \approx A'_q L^{\tilde{\psi}_v(q)} \tag{1.3}$$

with

$$\tilde{\psi}_v(q) = \tilde{\psi}(q) - 2q\tilde{\psi}(1) \tag{1.4}$$

The distribution of i_b (or of i'_b) is considered multifractal when $\tilde{\psi}(q)$ and A_q [or $\tilde{\psi}_v(q)$ and A'_q] become L-independent for large L, and when $\tilde{\psi}(q)$ is not linear in q. Recent work showed[3,4] that there exists a negative threshold q_c such that "conventional" multifractality (as defined above) is broken for $q < q_c < 0$. In this regime the moments are dominated by very small currents, which decay to zero faster than a power law in L. This results in the divergence of $\ln \hat{M}_q/\ln L$ (and $\ln \hat{M}^v_q/\ln L$) as $L \to \infty$.

Hierarchical structures imitate many geometrical features of the percolating clusters at p_c. Indeed, the moments $\hat{M}_q$ and $\hat{M}^v_q$ on such structures obey Eqs. (1.2) and (1.3), respectively for *all* q (thus missing the anomalous behavior at negative q). The multiplicative hierarchy of the currents on these structures implies that the distribution of $y_b = |\ln i_b|$, $n(y_b)$, is much simpler than that of i_b. In fact, $n(y_b)$ turns out to be *unifractal*,[2,4] and it depends on y_b only through the scaled combination $y_b/\ln L$. This leads to a simple behavior of the normalized moments

$$\mu_k = \sum_b y^k_b / N_{\mathrm{BB}} \tag{1.5}$$

where $N_{BB} = \hat{M}_0$ is the number of backbone bonds, which have $i_b \neq 0$. On hierarchical lattices, μ_k scales asymptotically as $(\ln L)^k$.[4]

Since hierarchical lattices miss the singular behavior of $\hat{M}_q$ for $q < q_c$, it is not clear *a priori* if the above results for $n(y_b)$ and μ_k survive for real percolating resistor networks. This led to detailed recent studies of $n(y_b)$ and μ_k on the latter structures.[4–6] In the absence of a theory, μ_k was first fitted to the generalized form[5]

$$\mu_k \approx B_k(\ln L)^{\beta(k)} \tag{1.6}$$

For three-dimensional percolating networks, $\beta(k)$ was fitted[5] to a straight line, $\beta(k) = ck$, and c was found to be between 1 and 1.15. Another interesting result was that $n(y_b)$ turned out to be very close to linear in y_b for large y_b (small i_b). The present paper reports on similar analyses of two-dimensional percolating resistor networks, using a modified transfer matrix method.[7] It should be emphasized that this transfer matrix method is essentially exact (apart from small roundoff errors). This is crucial for finding the correct small currents. For these currents, we expect to do much better than the competing methods (see, e.g., ref. 8).

In parallel to our numerical work, Aharony *et al.* (ABH)[4] developed a theory for $n(y_b)$ and μ_k. They found that for large L,

$$\mu_k \approx (\alpha_0 \ln L)^k \{1 + k[C_1 + \tfrac{1}{2} D_1(k-1)](\ln L)^{-1} + O[(\ln L)^{-2}]\} \tag{1.7}$$

Therefore, they expect that asymptotically $\beta(k) = k$. However, the finite-size corrections are of relative order $k(k-1)/\ln L$, which is not at all small for realistic values of L. These predictions found some support in recent series expansion results.[6]

After giving a brief description of our numerical method in Section 2, we present our results for $\hat{M}_q$, $\hat{M}_q^v$, $n(y_b)$, and μ_k in Section 3. Section 4 then contains a critical discussion of these results, and our final conclusions.

2. THE NUMERICAL METHOD

As mentioned, we considered a percolating square lattice resistor network of size $L \times L$ with either unit-current or unit-voltage boundary conditions. Data were collected for $M = 5000$, 5000, 5000, 3275, 1433, 735, 416, 509, 325, and 329 samples at $L = 6$, 10, 20, 30, 40, 50, 60, 70, 80, and 90, respectively.

The currents in all the bonds of the network were obtained by a modified transfer matrix method,[7] which uses two independently derived admittance matrices that characterize the left- and right-hand sides of the network. In this method we calculated all the bond currents in a given cross section of the network, by solving a set of linear equations obtained by equating the currents and voltages from the two sides of the network. The bond currents obtained in this way are extremely accurate, as required for the logarithms-of-the-currents distribution, without having to eliminate the dangling ends. (We chose not to eliminate these bonds, because we also wanted to calculate the distribution of the voltage drops over insulating bonds; see the end of Section 3. In a few cases we did "burn" the dangling ends; this neither saved on the total computer time nor increased the accuracy. In fact, the only errors in this method, which does not rely on iterations and is essentially exact, are due to roundoff.

In the analysis of the logarithms-of-the-currents distribution, it is important to avoid even those roundoff errors, since the smallest currents i give the largest values of $|\ln(i)|$. For large values of L ($L > 40$) there appear some very small currents that cannot be clearly distinguished from spurious values that appear, as a result of roundoff errors, in conducting bonds where the current should vanish (e.g., in dangling clusters or on exactly balanced elements, such as the Wheatstone bridge). As we explain below, we discarded the main contribution of these spurious currents by introducing a lower cutoff. We found that the moments which were studied did not depend on the exact position of this cutoff. Therefore, the spurious currents are irrelevant for these moments.

The data on the distribution of the logarithms-of-the-currents were collected from M sample systems in $N = 1000$ bins equally spaced in $\ln i$ between $\ln(i_{\min})$ and $\ln(i_{\max})$. The values of $i_{\min}$ and $i_{\max}$ were chosen from the analysis of preliminary results, so that all the calculated currents fell between these values. Initially we chose $i_{\min} = 10^{-12}$, $i_{\max} = 1$.

As a preliminary check, we accumulated the data for $|\ln i_b|^k = y_b^k$ on all the bonds with $i_b \neq 1$ within each particular sample and then averaged the results for μ_k over the M samples. This gave us some estimates for the statistical error bars. We also accumulated the data for y^k in *all* the samples and then normalized the result by the total number of bonds. The results of these two calculations were close to each other, indicating a large degree of self-averaging.

Having plotted the data for the full distribution $n(y)$, we noted that it decreased monotonically for large y (small i), reached a minimum, and then increased (see the large-y part of Fig. 1). We associated this increase with spurious currents, resulting from roundoff errors. We thus redefined $i_{\min}$ as the current that corresponds to this minimum, and discarded all smaller currents as spurious.[5, 9] Finally, we used the remaining distributions $\hat{n}(i)$ and $n(y)$ to obtain the moments of i and of $y = |\ln i|$, as presented below.

3. RESULTS

Figure 1 shows $n(y)$ for $L = 90$, for unit-voltage boundary conditions, including spurious currents, whose contributions dominate the distribution for $y \gtrsim 20$. That part of the distribution was discarded in the calculation of the moments μ_k.

The most pronounced feature of Fig. 1 is the linear dependence of $\ln n(y)$ on $y = |\ln i|$ over the range $10^{-8} < i < 10^{-3}$. Fitting this part of the curve to straight lines for several sizes and extrapolating their slopes to $L \to \infty$ gave

$$\ln n(|\ln i|) \approx -\gamma \, |\ln i| \tag{3.1}$$

with $\gamma = 0.5 \pm 0.1$. As discussed by ABH,[4] this linear dependence indicates a breakdown of multifractality in the moments $\hat{M}_q$ for $q \leqslant q_c$. They also argued that the slope γ yields an upper bound on q_c,

$$q_c \precsim -\tfrac{1}{2}\gamma \simeq -0.25 \tag{3.2}$$

For $q < q_c$, the moment $\hat{M}_q$ is dominated by the lower bound in the integral (1.1), and it behaves as $|i_{\min}|^{2q+\gamma}$.

We next turn to the moments of the logarithms of the current, μ_k. Applying a least-squares fit to the straight line

$$\ln \mu_k = \ln B_k + \beta(k) \ln(\ln L) \tag{3.3}$$

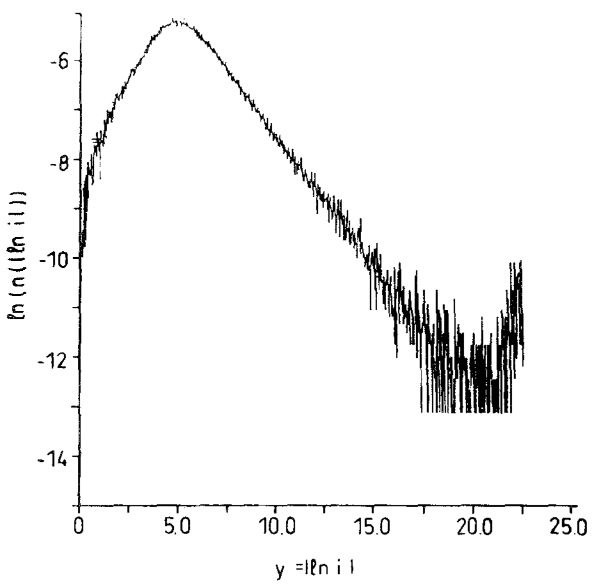

Fig. 1. Logarithm-of-the-current distribution $n(y)$ versus $y = |\ln i|$ for $L = 90$.

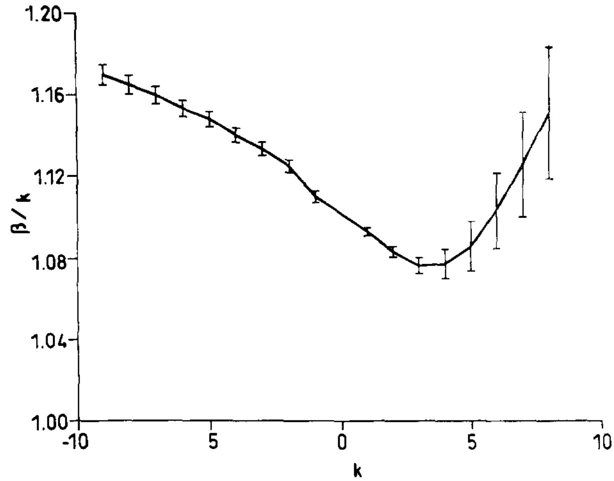

Fig. 2. Plot of β/k versus k when a unit voltage is applied between the busbars.

we deduced values for $\beta(k)$ and B_k, as plotted in Figs. 2 and 3. The error bars in these figures are only statistical. Even with these small error bars, the values of $\beta(k)/k$ are quite close to unity (note the large scale of Fig. 2). In fact, the observed slopes of the curves of $\ln \mu_k$ versus $\ln(\ln L)$ increased slightly at large L for $k > 1$. As we discuss below, this increase [which

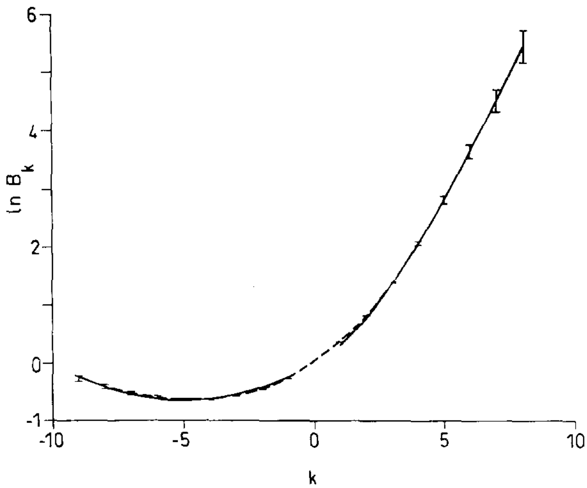

Fig. 3. Plot of $\ln(B_k)$ versus k when a unit voltage is applied between the busbars. The line drawn through the points can be represented by $k^{1.38}$ for $k > 0$. A fit with a second-order polynomial is shown for negative values of k.

Table I. Exponents and Amplitudes of the Moments of the Current Distributions

q	$\ln A'_q$	$\ln A_q$	$\tilde{\psi}_v(q)$	$\tilde{\psi}(q)$
1	0.114 ± 0.009	0.04 ± 0.02	-0.980 ± 0.003	0.977 ± 0.007
2	0.12 ± 0.02	0.03 ± 0.04	-3.121 ± 0.006	0.81 ± 0.01
3	0.26 ± 0.04	0.02 ± 0.05	-5.14 ± 0.01	0.77 ± 0.01
4	0.62 ± 0.07	0.01 ± 0.05	-7.12 ± 0.01	0.75 ± 0.01

causes an increase in the measured average slope $\beta(k)$] may be due to a systematic error concerning spurious currents. The true $\beta(k)$ values may thus be smaller.

A power law fit to $\ln B_k$ of Eq. (1.6) for $k > 0$ yields $\ln B_k \sim k^{1.4}$ (see Fig. 3). The data for $k < 0$ were fitted by a second-order polynomial. If we wish to fit all the points, with k both positive and negative, by a single analytic expression, we need a fourth-order polynomial. However, the data in Fig. 3 for $k > 0$ may also be fitted by a linear curve, as predicted from Eq. (1.7), i.e., $B_k \simeq \alpha_0^k$, with corrections which increase with k.

We next turn to moments of the current, $\hat{M}_q$. Least-square fits of straight lines for $\ln \hat{M}_q$ and $\ln \hat{M}_q^v$ versus $\ln L$ [Eqs. (1.2) and (1.3)] yielded the results listed in Table I. We note that A_q seems to depend very weakly on q, while A'_q shows a stronger dependence on q. The measured values of $\tilde{\psi}_v(q)$ and $\tilde{\psi}(q)$ satisfy Eq. (1.4). However, if this equation is used to evaluate $\tilde{\psi}_v(q)$ from $\tilde{\psi}(q)$, then one encounters larger errors compared to the direct measurement of $\tilde{\psi}_v(q)$. These errors arise from the term $2q\tilde{\psi}(1)$.

In Table II we compare our results for the exponents $\tilde{\psi}(q)$ with results from other simulations and also from series expansions. It is clear from this table that our results not only are consistent with previous calculations,

Table II. Comparison of Our Results for $\tilde{\psi}_v(q)$ and $\tilde{\psi}(q)$ with Results from Other Simulations and from Series Expansions

Source	$\tilde{\psi}_v(2)$	$\tilde{\psi}(1)$	$\tilde{\psi}(2)$	$\tilde{\psi}(3)$	$\tilde{\psi}(4)$
Our results	-3.121 ± 0.006	0.977 ± 0.007	0.81 ± 0.01	0.77 ± 0.01	0.75 ± 0.01
Series[3]	—	—	0.825 ± 0.06	0.78 ± 0.06	0.765 ± 0.06
Simulation[10]	-3.057 ± 0.03	0.973 ± 0.05	—	—	—
Simulation[11]	—	0.982 ± 0.004	0.818 ± 0.009	0.773 ± 0.01	—
Simulation[2]	-3.12 ± 0.02	—	—	—	—
Simulation[12]	—	0.978 ± 0.01	0.81 ± 0.02	0.74 ± 0.02	0.73 ± 0.02
Simulation[13]	—	0.9745 ± 0.0015	—	—	—
Simulation[14]	—	1.0035 ± 0.0096	0.857 ± 0.013	0.816 ± 0.014	0.803 ± 0.015

but are also very precise. We take this to be an indication of the reliability and precision of our simulations and of our other results as well.

We also evaluated the voltages v across insulating bonds that separate pairs of adjacent backbone sites. We then calculated moments of the distribution of $\ln v$, $\bar{n}(|\ln v|)$. These moments, as well as the distribution itself, are more susceptible to errors from the spurious currents than are the moments of the distribution $n(y)$. Assuming that we can still apply an equation of the form (1.6) to describe the new moments, we obtain $\bar{\beta}(k)/k \simeq 1.13 \pm 0.06$. This value is close to the effective one obtained from $n(y)$ (without taking into account finite-size corrections), which may suggest that the two distributions have similar scaling properties.

4. DISCUSSION AND CONCLUSIONS

As mentioned above, our results may suffer from two main problems. The first one concerns the spurious small currents due to roundoff errors. Even though we eliminate most of these by shifting $i_{\min}$ from 10^{-12} to the minimum in $n(y)$ (see Fig. 1), some spurious currents may still remain. These may cause a misleading increase in μ_k for large L, which grows larger with increasing $k > 1$. This may be one source for the observed increase in $\beta(k)/k$ in this range (see Fig. 2). Since results are not sensitive to the exact choice of the cutoff, we feel that this problem is not severe.

A second major problem concerns finite-size effects. It is interesting to note that Eq. (1.7) predicts that the effective slope $\beta_{\text{eff}}(k)$ should behave as

$$\beta_{\text{eff}}(k) = \frac{\partial \ln \mu_k}{\partial \ln \ln L} = k \left\{ 1 - \left[C_1 + \frac{1}{2} D_1 (k-1) \right] (\ln L)^{-1} + O[(\ln L)^{-2}] \right\} \tag{4.1}$$

Therefore, to order $(\ln L)^{-1}$, $\beta_{\text{eff}}(k)/k$ should be *linear* in k. Since Fig. 2 exhibits a nonmonotonic dependence of $\beta_{\text{eff}}(k)/k$, it is clear that one needs higher order terms, e.g., of order[4] $k(k-1)(k-2)(\ln L)^{-2}$, in order to fit the data. This is a direct consequence of the fact that for our data, $k/\ln L$ became of order unity for $|k| \sim 4$. Since the data were forced into the form of Eq. (3.3) with the nonmonotonic effective exponent $\beta_{\text{eff}}(k)$, this resulted in a similar nonmonotonic effective intercept $\ln B_k$, as seen from Fig. 3. Larger samples are needed for a systematic comparison with the corrections in Eq. (1.7). However, the closeness of our $\beta(k)/k$ to unity is encouraging.

A third problem, not mentioned yet in this paper, concerns the finite resolution in binning the data in Fig. 1. As shown recently,[15] the apparently linear part of the curve may result from or be affected by the resolution. Future work should study the related effects here.

In conclusion, very precise values of $\tilde{\psi}(q)$ were obtained and our numerical results for $\beta(k)$ seem to be consistent with the theoretical predictions of ABH. However, it is clear that better statistics, larger values of L, more systematic inclusion of the finite-size corrections, better ways to eliminate roundoff errors, and a systematic study of finite-resolution effects are necessary in order to be more conclusive.

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